IIR Digital Filter Design

Standard approach

(1) Convert the digital filter specifications into an analogue prototype lowpass filter specifications

(2) Determine the analogue lowpass filter transfer function $H_a(s)$

(3) Transform $H_a(s)$ by replacing the complex variable to the digital transfer function

$$G(z)$$

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- This approach has been widely used for the following reasons:

  (1) Analogue approximation techniques are highly advanced

  (2) They usually yield closed-form solutions

  (3) Extensive tables are available for analogue filter design

  (4) Very often applications require digital simulation of analogue systems

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- Let an analogue transfer function be

$$H_a(s) = \frac{P_a(s)}{D_a(s)}$$

  where the subscript "*a*" indicates the analogue domain

- A digital transfer function derived from this is denoted as

$$G(z) = \frac{P(z)}{D(z)}$$

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- Basic idea behind the conversion of $H_a(s)$ into $G(z)$ is to apply a mapping from the *s*-domain to the *z*-domain so that essential properties of the analogue frequency response are preserved
- Thus mapping function should be such that
  - Imaginary $(j\Omega)$ axis in the *s*-plane be mapped onto the unit circle of the *z*-plane
  - A stable analogue transfer function be mapped into a stable digital transfer function

IIR Digital Filter: The bilinear transformation

- To obtain $G(z)$ replace s by $f(z)$ in $H(s)$
- Start with requirements on $G(z)$

<u>$G(z)$</u>	<u>Available $H(s)$</u>
Stable	Stable
Real and Rational in z	Real and Rational in s
Order n	Order n
L.P. (lowpass) cutoff Ω_c	L.P. cutoff $\omega_c T$

IIR Digital Filter

- Hence $f(z)$ is real and rational in *z* of order <u>one</u>
- i.e. $$f(z) = \frac{az+b}{cz+d}$$
- For LP to LP transformation we require

$$s = 0 \quad \rightarrow \quad z = 1 \quad f(1) = 0 \rightarrow a + b = 0$$

$$s = \pm j\infty \rightarrow \quad z = -1 \quad f(-1) = \pm j\infty \rightarrow c - d = 0$$

- Thus $$f(z) = \left(\frac{a}{c}\right) \cdot \frac{z-1}{z+1}$$

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- The quantity $\left(\frac{a}{c}\right)$ is fixed from $\omega_c T \leftrightarrow \Omega_c$
- ie on $C : |z| = 1$ $\quad f(z)|_c = \left(\frac{a}{c}\right) . j \tan \frac{\omega T}{2}$
- Or

$$j\Omega_c = \left(\frac{a}{c}\right) . j \tan \frac{\omega_c T}{2}$$

- and

$$s = \left(\frac{\Omega_c}{\tan\left(\frac{\omega_c T}{2}\right)} \right) . \frac{1 - z^{-1}}{1 + z^{-1}}$$

Bilinear Transformation

- Transformation is unaffected by scaling. Consider inverse transformation with scale factor equal to unity

$$z = \frac{1+s}{1-s}$$

- For

$$s = \sigma_o + j\Omega_o$$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

- and so

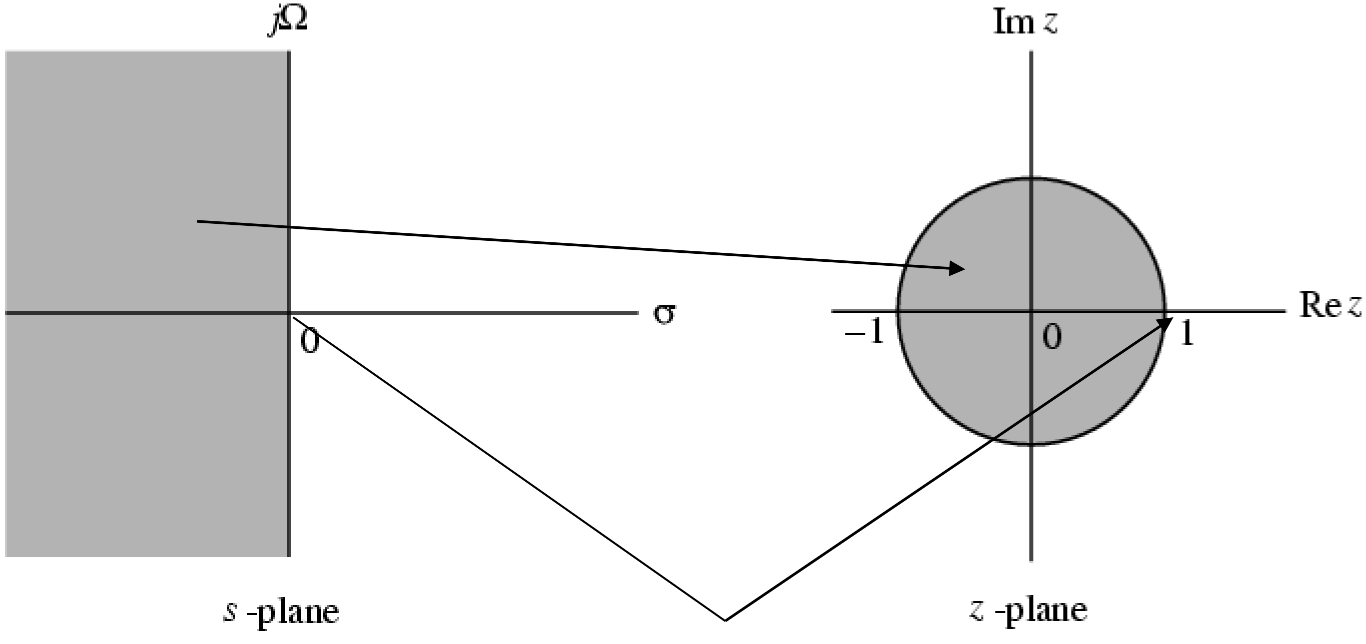
$$\sigma_o = 0 \rightarrow |z| = 1$$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

Bilinear Transformation

- Mapping of *s*-plane into the *z*-plane



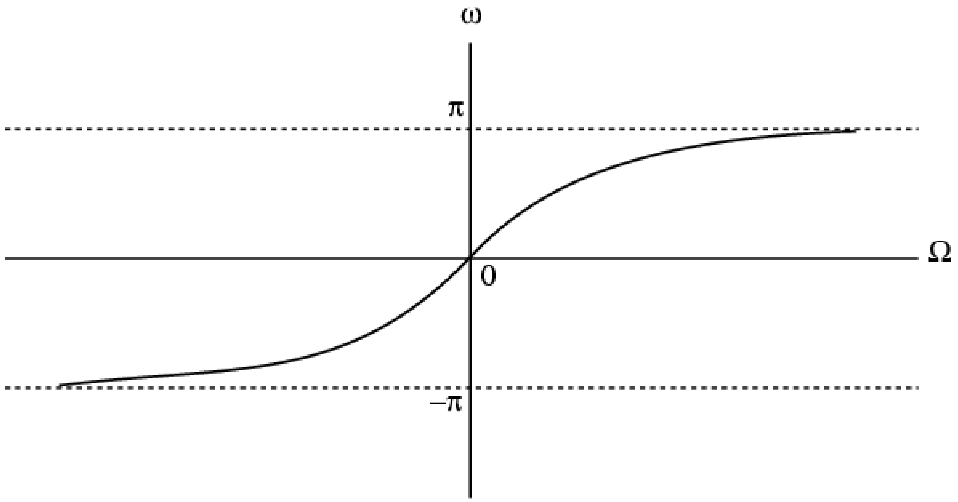
Bilinear Transformation

- For $z = e^{j\omega}$ with *unity scalar* we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j\tan(\omega/2)$$

or

$$\Omega = \tan(\omega/2)$$



Bilinear Transformation

- Mapping is highly nonlinear
- Complete negative imaginary axis in the *s*-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle in the *z*-plane from $z = -1$ to $z = 1$
- Complete positive imaginary axis in the *s*-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the *z*-plane from $z = 1$ to $z = -1$

Spectral Transformations

- To transform $G_L(z)$ a given lowpass transfer function to another transfer function $G_D(\hat{z})$ that may be a lowpass, highpass, bandpass or bandstop filter (solutions given by Constantinides)
- z^{-1} has been used to denote the unit delay in the prototype lowpass filter $G_L(z)$ and $\hat{z}^{-1}$ to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion

Spectral Transformations

- Unit circles in *z*- and $\hat{z}$ -planes defined by

$$z = e^{j\omega}, \qquad \hat{z} = e^{j\hat{\omega}}$$

- Transformation from *z*-domain to $\hat{z}$ -domain given by

$$z = F(\hat{z})$$

- Then

$$G_D(\hat{z}) = G_L\{F(\hat{z})\}$$